Fourier Analysis 03-02

Review

Thm. Let $f \in \mathcal{R}$. Suppose f is differentiable at x_0. Then $\lim_{N\to\infty} S_N f(x_0) = f(x_0)$.

Thm (Dirichlet-Dini Criterion)

Let $f \in \mathcal{R}$. Suppose $\exists\, u \in \mathbb{R}$ such that

(Dini condition) $$\int_0^{\pi} \left| \frac{f(x_0+y)+f(x_0-y)}{2} - u \right| \frac{dy}{y} < \infty.$$

Then $S_N f(x_0) \to u$ as $N \to +\infty$.

Cor. If $f \in \mathcal{R}$, f is Hölder cts at x_0, then

$$S_N f(x_0) \to f(x_0) \quad \text{as } N \to \infty.$$

§ 3.5 A cts function with diverging Fourier series

Let us start from a special function:

$$f(x) = \begin{cases} i(\pi - x) & \text{if } 0 \leq x \leq \pi \\ i(-\pi - x) & \text{if } -\pi \leq x < 0 \end{cases}$$

It is an odd function except at $x=0$.

It has the following Fourier series

$$f(x) \sim \sum_{n \neq 0} \frac{1}{n} e^{inx} \quad \text{on } [-\pi, \pi].$$

Write for $N \in \mathbb{N}$,

$$f_N(x) = \sum_{\substack{-N \leq n \leq N \\ n \neq 0}} \frac{1}{n} e^{inx}$$

$$\tilde{f}_N(x) = \sum_{n=-N}^{-1} \frac{1}{n} e^{inx}.$$

<u>Lemma 1</u>. (1) $\exists\, M > 0$ such that

$$|f_N(x)| \leq M \quad \text{for all } N \in \mathbb{N},\ x \in [-\pi, \pi]$$

(2) $|\tilde{f}_N(0)| \geq \log N$.

Pf. We first prove (1). Consider the Abel mean of f,

$$A_r(f)(x) = \sum_{n\neq 0} \frac{r^{|n|}}{n} e^{inx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x-y) f(y)\, dy$$

where $0 \leq r < 1$.

$$|A_r(f)(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x-y)\,|f(y)|\, dy$$

$$\leq \sup_{z\in[-\pi,\pi]} |f(z)| \; \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x-y)\, dy$$

$$= \|f\|_\infty < \infty$$

Notice that

$$|f_N(x) - A_r(f)(x)| \leq \Big| f_N(x) - \sum_{0<|n|\leq N} \frac{r^{|n|}}{n} e^{inx} \Big|$$

$$+ \Big| \sum_{|n|\geq N+1} \frac{r^{|n|}}{n} e^{inx} \Big|$$

$$= \Big| \sum_{0<|n|\leq N} \frac{(1-r^{|n|})}{n} e^{inx} \Big|$$

$$+ \Big| \sum_{|n|\geq N+1} \frac{r^{|n|}}{n} e^{inx} \Big|$$

$$\leq \sum_{0<|n|\leq N} \frac{1-r^{|n|}}{|n|} + \sum_{|n|\geq N+1} \frac{r^{|n|}}{|n|}$$

$$= 2\sum_{n=1}^{N} \frac{1-r^n}{n} + 2\cdot \sum_{n\geq N+1} \frac{r^n}{n}$$

(Notice that $\displaystyle \frac{1-r^n}{n} = \frac{(1-r)(1+r+\cdots+r^{n-1})}{n} < 1-r$

$$\sum_{n\geq N+1} \frac{r^n}{n} \leq \sum_{n\geq N+1} \frac{r^n}{N}$$

$$= \frac{r^{N+1}}{N(1-r)} \quad)$$

Hence

$$|f_N(x) - A_r(f)(x)| \leq 2\cdot N(1-r) + 2\cdot \frac{r^{N+1}}{N(1-r)}$$

$$\leq 2\cdot N(1-r) + \frac{2}{N(1-r)}$$

Taking $r = 1-\frac{1}{N}$, then $N(1-r)=1$ so

$$|f_N(x) - A_r(f)(x)| \leq 4$$

Hence $\quad |f_N(x)| \leq |A_r(f)(x)| + 4$

$$\leq \|f\|_\infty + 4 \leq 2\pi + 4.$$

This proves (1).

Now $\quad |\widehat{f_N}(0)| = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$

$$\geq \sum_{k=1}^{N} \int_k^{k+1} \frac{1}{x}\, dx$$

$$\geq \sum_{k=1}^{N} (\log(k+1) - \log k) = \log(N+1) > \log N.$$

$\square$

Now for each $N \in \mathbb{N}$, define

$$P_N(x) = e^{i2Nx} \cdot f_N(x) = \sum_{\substack{n=N \\ n \neq 2N}}^{3N} \frac{1}{n-2N} e^{inx}$$

$$\tilde{P}_N(x) = e^{i2Nx} \tilde{f}_N(x) = \sum_{n=N}^{2N-1} \frac{1}{n-2N} e^{inx}$$

Define a sequence of integers $(N_k)_{k=1}^{\infty}$ and
a sequence of positive numbers $(\alpha_k)_{k=1}^{\infty}$ such that

(i) $N_{k+1} > 3N_k$ for all k

(ii) $\sum_{k=1}^{\infty} \alpha_k < \infty$.

(iii) $\alpha_k \log N_k \to \infty$ as $k \to \infty$.

(e.g. we can take $N_k = 4^{4^k}$, $\alpha_k = 3^{-k}$.)

Define

$$g(x) = \sum_{k=1}^{\infty} \alpha_k \cdot P_{N_k}(x)$$

Since $|P_{N_k}(x)| = |f_{N_k}(x)| \leq M$

So the series converges absolutely and g is cts.

We would like to show that

$$S_N g(0) \not\to g(0).$$

__Lemma 2__: Let $n \in \mathbb{Z}$. Then

$$\hat{g}(n) = \begin{cases} \dfrac{1}{n-2N_k} & \text{if } N_k \leq n \leq 3N_k \text{ but } n \neq 2N_k \\ 0 & \text{otherwise} \end{cases}$$